From QFT to 2-d supersymmetric TFT : Autoduality, Mirror symmetry between maths and physics : Atiyah-Witten work : an "instantonic Geometry"

Ph. Durand : *Conservatoire national des arts et métiers Paris*

Villa de Leyva July 4 – 22 2011

1 Introduction

The introduction of supersymmetric field theories in two dimensions, has contributed to strengthen the links between geometry and physics.The beginning is Atiyah work, mainly index theorem : a smooth form of Riemann-Roch and the applications on physics. With Segal he give some axioms of topological field theory. TQFT find a concrete realisation in the supersymmetric world which adapt the Noether symmetry by adding supersymmetric variables.that helped extend the BRST formalism for supersymmetric topological field theories . Calculations of correlation functions in which the points are replaced by cohomology classes, lead to define spaces of instantons (moduli) whose dimension calculated by the index theorem is void and has lead to stage enumerative geometry. From 1985, Donalson, Gromov and Witten have understood the importance of the concept of instantons defined in various moduli spaces, and permitted a better understanding of the geometry of four, six, teen dimensions and concepts of complex curves and surfaces.The presentation will provide a few steps of this intellectual journey and relevance of the coupling between geometry and physics.

2 From QFT to TQFT

A quantum field theory can be seen in through a classic action was quantified from the path integral. We can then define the correlation functions dependent on observable dependent on selected points on the source space.These functions depend, for example, the metric that is chosen on the target space.The Polyakov action in string theory is much better than that of Nambu goto because it allows easily obtained through the geometry, as well the topological Yang-Mills action is independent of the metric.It appears natural to define a field theory "topological" free of sophisticated structures like differentiability, metric. In such a theory correlation functions depend only on topological objects such as class (co) homology

2.1 TQFT : From Atiyah-Segal program

The axiomatic given by Atiyah is based on ***CFT*** in two dimension given by Segal. the idea : each riemann surfaces can be divided like ***"Lego game"***. The most famous piece is the pair of pants.A TQFT of dimension $n+1$ associated to each n dimensional manifold X an vector space $V(X)$.
A cobordism is data (M, X, Y), with $\partial M = (-X) \sqcup (Y)$ and $dim M = n+1$.TQFT must satisfy some axiom : Naturality, fonctoriality, normalization, multiplicativity, symmetry .

2.2 TQFT : From Witten view

Witten approach is based in three words the ***path integral***, and correlation functions ***supersymmetry*** and ***localisation***.
A ***correlation functions*** in $TQFT$ is given by
$\langle \mathcal{O}_{\mu_1}, \mathcal{O}_{\mu_2}, ..., \mathcal{O}_{\mu_n} \rangle = \int [\mathcal{D}_{\phi i}] \mathcal{O}_{\mu_1}(\phi_i), \mathcal{O}_{\mu_2}(\phi_i), ..., \mathcal{O}_{\mu_n}(\phi_i) exp(-S(\phi))$

Topological meaning requires : $\frac{\delta}{\delta g_{\mu\nu}} \langle \mathcal{O}_{\mu_1}, \mathcal{O}_{\mu_2}, ..., \mathcal{O}_{\mu_n} \rangle = 0$

there is a ***supersymmetric*** operator Q that checks : $\delta S = -i\epsilon\{Q, S\} = 0$.
The integral is ***localized*** on the space of instantons (zero-modes) and is zero for other configurations.

2.3 Supersymmetric quantum mechanic

Now we study a supersymmetric field theory in ***one dimension***. This is the model of ***supersymmetric quantum mechanics*** which has allowed Witten to give a new proof of the ***index theorem***. We considère the lagrangian :
$L = \frac{\dot{x}^2}{2} - \frac{h'(x)^2}{2} + i(\bar{\psi}\dot{\psi} - \dot{\bar{\psi}}\psi) - h"(x)\bar{\psi}\psi$.
$\psi = \psi_1 + i\psi_2$
$\bar{\psi} = \psi_1 - i\psi_2$

Let $\pi = \frac{\partial L}{\partial \dot{\psi}} = i\bar{\psi}$, $p = \frac{\partial L}{\partial \dot{x}} = \dot{x}$ the conjugate moments.

Let ***supersymmetric*** relations
$\delta_\epsilon x = \epsilon\bar{\psi} - \bar{\epsilon}\psi$
$\delta_\epsilon \psi = \epsilon(i\dot{x} + h'(x))$ $\epsilon = \epsilon_1 + i\epsilon_2$
$\delta_\epsilon \bar{\psi} = \epsilon(i\dot{x} + h'(x))$ We can show :
$\delta_\epsilon S = \int \delta L dt = \int \frac{d}{dt} L dt = 0$

The two operators of supersymmetry, are associated ***supercharge*** Q, $\bar{Q}$ with $Q^2 = \bar{Q}^2 = 0$ and we deduce an elliptic complex :

$\mathcal{H}_F \xrightarrow{Q,\bar{Q}} \mathcal{H}_B \xrightarrow{Q,\bar{Q}} \mathcal{H}_F \xrightarrow{Q,\bar{Q}} ....$

In hamiltonian formalism, $\{Q, \bar{Q}\} = 2H.SQM$ compactified on S^1 give :

$Tr(-1)^F e^{-\beta H} = dim\mathcal{H}^B_{(0)} - dim\mathcal{H}^F_{(0)}$ with F ***fermion number***.

The supertrace giving the index, expressed by :

$Tr(-1)^F e^{-\beta H} = \int_{periodicBd} \mathcal{D}X\mathcal{D}\psi\mathcal{D}\overline{\psi}e^{-S}$

The fundamental result is that ***only time-independent contribute*** : that reduce calculation to 0-dim TQFT :
$\mathcal{Z} = Tr(-1)^F e^{-\beta H} = \sum_{h'(x_c)=0} \sqrt{\pi}\frac{h"(x_c)}{|h"(x_c)|}$

3 TQFT and Lower dimensional geometry

The geometry of one and two dimensions is well know. Henri Poincare invented algebraic topology, and the Betti numbers that generalizes the invariant Euler. It can be shown that in dimension smaller than three, the category of ***topological*** varieties is identical to that of ***differentiable*** manifolds. (1923 Kerékjarko for dimension 2, Moises, Bing (1950) for the dimension 3). However this result is false in dimension four (in 1956 Milnor shows that there are 27 structures on the differentiable sphere S_7.

3.1 Dimension two

In two dimensions, the classification of surfaces topological (so differentiable), is made by the ***Betti numbers***, and the ***orientability*** allows for example to distinguish between torus and Klein bottle. In particular, it can easily demonstrate the Poincare conjecture in dimension 2 : any simply connected compact surface ($\pi_1(S) = 0$) is homeomorphic to the sphere and as H_1 is abelianised of π_1, the first Betti number is zero and the surface does not contain holes.

3.2 Dimension four

In four dimension, let X a compact a variety simply connected,we have $H_1(X) = 0$, so by Poincare duality, we also $H_3(X) = 0$, only $H_2(X)$ may be nonzero. One can intuit ***that two non-trivial cycles*** (areas), Σ_1, Σ_2 in general position in X ***contribute to the homology*** and thus provide topological information about it. is exactly what showed Freedman in 1982 Four dimensionnal manifolds X, are classified by the ***intersection quadratic form*** $H_2(X, \mathbb{Z}) \times H_2(X, \mathbb{Z}) \to \mathbb{Z}$.

If we denoted it , I_X we have for example :

1. $I_{S^4} = 0$: there is not non trivial cycles in the four sphere ($H_2(S^4, \mathbb{Z}) = 0$).
2. $I_{S^2 \times S^2}, = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$: $H_2(S^2 \times S^2, \mathbb{Z})= \mathbb{Z} \oplus \mathbb{Z}$: There is two cycles in general position $A = S^2 \times pt$, $B = pt \times S^2$ et $\langle A, B\rangle = \langle A, B\rangle = 1$, $\langle A, A\rangle = \langle B, B\rangle = 0$.
3. $I_{M\sharp N} = \begin{pmatrix} I_M & 0 \\ 0 & I_N \end{pmatrix}$:$H_2(M\sharp N, \mathbb{Z}) = H_2(M, \mathbb{Z}) \oplus H_2(M, \mathbb{Z})$, ($M\sharp N$ is connect sum of two four dimensional manifolds.

Theorem : Freedman (1982) A symply connected four manifold X with even intersection form, q belongs to a ***unique homeomorphism class***, while if $q$is odd, there are just ***two non-homeomorphic*** X with q as their intersection form.

Application : Poincaré theorem Take $X = S^4$ and S^4_h an ***homotopic deformation*** of X, intersection form of X is zero so even. S^4_h has the same cohomology class than X by ***the previous theorem***, so the same intersection class that prove Poincaré theorem.

<u>**Theorem : Donaldson (1983)**</u> A simply connected smooth four manifold, with ***positive definite intersection form*** is always diagonalisable with the property : ***each eigenvalues equal to one***.

3.3 Yang-Mills : Donaldson theory

There is two important numbers in four-manifolds X : the ***Pontriagin*** and the ***Euler character*** :
$p(X) = 3\sigma(X)\ e(X) = \chi(X)$.
Form in four-manifold split into ***Self dual : SD***, and ***anti self dual :ASD*** a connection is ASD if $F_A^+ = 0$ The main result his that ASD connection ***minimize*** Yang Mills action :

$S_{YM} = \frac{1}{2}\int_X F \wedge *F$. We can write :
$S_{YM} = \frac{1}{2}\int_X |F_A^+|^2 d\mu + 8\pi^2 k$

In the last expression, k is a ***topological invariant*** :
Chern-Weil theory give $k = -\frac{1}{8\pi^2}\int_X tr(F_A^2)$
Donaldson define the ***moduli space*** of ASD connections as follows :

$\mathcal{M}_{ASD} = \{[A] \in \mathcal{A}/\mathcal{G}/F_A^+ = 0\}$ usually in this theory $\mathcal{G} = SU(2)$.

3.4 Local model

It is important to have a ***local model*** of the moduli space of connection we must linearize near ASD connection : define ***tangent space*** $T_{[A]}\mathcal{M}_{ASD}$. That give the (virtual) ***dimension of moduli space*** :
There his elliptic complex let P a principal SU(2)-bundle on X named ***The instanton deformation complex*** :

$0 \to \Omega^0(X, adP) \xrightarrow{\nabla_A} \Omega^1(X, adP) \xrightarrow{P_+\nabla_A} \Omega^2_+(X, adP) \to 0$
We have :$H_A^1 = T_{[A]}\mathcal{M}_{ASD} = \frac{Kerp^+\nabla_A}{Im\nabla_A}$.

The index of this complex is the dimension of $\mathcal{M}_{ASD}$:
$dim\mathcal{M}_{ASD} = dimH_A^0 - dimH_A^1 + dimH_A^2$
Finally dimension of moduli space is given by :
$dim\mathcal{M}_{ASD} = -2p_1(V) - \frac{3}{2}(\chi + \sigma)$

3.5 Donaldson invariants

Let $\mathcal{A}^*$ irreductible connections, $\mathfrak{g}_E = adP$, $\mathcal{B}^* = \mathcal{A}^*/\mathcal{G}$
We define a new space $\mathcal{B}^* \times X$, where we can compute invariants :

$$
\begin{array}{ccccc}
\mathcal{A}^* \times_{\hat{\mathcal{G}}} \mathfrak{g}_E & \quad & \mathcal{A}^* \times \mathfrak{g}_E & \longrightarrow & \mathfrak{g}_E \\
\downarrow \pi & & \downarrow \pi & & \downarrow \pi \\
\mathcal{B}^* \times X & & \mathcal{A}^* \times X & \longrightarrow & X
\end{array}
$$

<u>**Donaldson map**</u>

We can define cohomology class in $H^4(\mathcal{B}^* \times X)$ we transform homology class on X in cohomology class on moduli space :
$\mu : H_i(X) \to H^{4-i}(\mathcal{B}^*)$

$$
\begin{array}{ccc}
x \in H_0(X) & \to & \mathcal{O}(x) \in H^4(\mathcal{M}_{ASD}) \\
\delta \in H_1(X) & \to & I_1(\delta) \in H^3(\mathcal{M}_{ASD}) \\
S \in H_2(X) & \to & I_2(S) \in H^2(\mathcal{M}_{ASD})
\end{array}
$$

3.6 Witten Donaldson invariants

If the departure in the supersymmetric formalisme "a la Witten" is the "topological" observable : $\mathcal{O} = Tr(\phi^2)$, we have ***descent equations*** :

$\mathcal{O}^{(1)} = tr(\frac{1}{\sqrt{2}}\phi\psi_\mu)dx^\mu$
$\mathcal{O}^{(2)} = -\frac{1}{2}tr(\frac{1}{\sqrt{2}}\phi F_{\mu\nu} - \frac{1}{4}\psi_\mu\psi_\nu)dx^\mu \wedge dx^\nu$
$\mathcal{O}^{(3)} = -\frac{1}{8}tr(\psi_\lambda F_{\mu\nu})dx^\lambda \wedge dx^\mu \wedge dx^\nu$
$\mathcal{O}^{(4)} = \frac{1}{32}tr(F_{\lambda\tau}F_{\mu\nu})dx^\lambda \wedge dx^\tau \wedge dx^\mu \wedge dx^\nu$

From Witten to Donaldson We have :

$I_1(\delta) = \int_{\delta \in H_1(X)} \mathcal{O}^{(1)}$
$I_2(S) = \int_{S \in H_2(X)} \mathcal{O}^{(2)}$

We can show that these observable are compatible with Donaldson map result.

3.7 Ten dimension : String theory

The same formalism developped by Witten in three, and four dimension hold in ten dimensional meaning. You need to choose adapted sigma-model, for example the first twist give "A"-model with : lagrangian :
L, the supersymmetric lagrangian of a super-string is given by :

$$L = 2t\int_\Sigma(\frac{1}{2}g_{IJ}\partial_z\phi^I\partial_{\bar{z}}\phi^J)d^2z + 2t\int_\Sigma(i\psi_z^{\bar{i}}D_{\bar{z}}\chi^i g_{\bar{i}i} + i\psi_z^i D_z\chi^{\bar{i}}g_{\bar{i}i} - R_{i\bar{i}j\bar{j}}\psi_{\bar{z}}^i\psi_z^{\bar{i}}\chi^j\chi^{\bar{j}})d^2z$$

We have supersymmetric transformation preserving action :

$$\begin{array}{cc} \delta x^I = \eta\chi^I & \delta\chi^I = 0 \\ \delta\psi_{\bar{z}}^i = \eta\partial_{\bar{z}}\phi_i & \delta\psi_z^{\bar{i}} = \eta\partial_z\overline{\phi_i} \end{array}$$

If $\delta\psi_{\bar{z}}^i = \delta\psi_z^{\bar{i}} = 0$, we recognize the conditions of Cauchy-Riemann ! : The instantons of this model are curves "minimum energy" according to Gromov : <u>holomorphic curves</u>

3.8 (Virtual) dimension of space of holomorphic curves

From the two previous formula (R.R. R.R. for curve and parametric curve, we can deduce the virtual dimension the moduli space of holomorphic curves. For this we can reason using the exact sequences :
$0 \rightarrow T_\Sigma \rightarrow f^*T_X \rightarrow N_{\Sigma/X} \rightarrow 0$
For details see [Pandharipande]
Dimension of $\mathcal{M}_g(X)$ By combining the two previous forms of the Riemann-Roch formulawe obtain the dimension of $\mathcal{M}_g(X)$:
$dim_{virt}\overline{\mathcal{M}}_{g,n}(X,\beta) = (dim X)(1-g) + \int_{f_*(\Sigma)} c_1(TX) + 3g - 3 + n$
We could find directly this result in symplectic case (manifolds that are treated are Kähler, hence symplectic) relying on the index of a Fredholm operator for an elliptic complex adhoc.

3.9 Gromov-Witten invariant

The notation [.] denote the fundamental class in $H_k(\overline{\mathcal{M}}_{g,n}(X,\beta),\mathbb{Q})$: We can now properly define the Gromov-Witten invariants.
Indeed, if $[\omega_1], ..., [\omega_n]$ are cohomology class in $H^*_{DR}(X)$ such that :
$\Sigma_{i=1}^n deg[\omega_i] = k$
"integration on moduli space " will then be a non-zero number, So we can expect count something.

Is called Gromov-Witten invariant the quantity :
$< [\omega_1], ...[\omega_n] >_\beta = \int_{[\overline{\mathcal{M}}_{g,n}(X,\beta)]} ev^*([\omega_1]) \wedge ... \wedge ev^*([\omega_n])$

In this script we used an evaluation map :

$$\begin{array}{rclcc} ev_i : \overline{\mathcal{M}}_{g,n}(X,\beta) \to X & : & (\Sigma, x_1, ..., x_n, \varphi) & \mapsto & \varphi(x_i) \\ \mathrm{ev}_i^* : H^*(X) \to H^*(\overline{\mathcal{M}}_{g,n}(X,\beta)) & : & [\omega_i] & \mapsto & ev_i^*([\omega_i]) \end{array}$$

This is completely analogous to how to obtain the invariants of Donaldson.

4 Conclusion : Set of rules for TQFT

In the topological theory proposed by Witten we can identify the next steps :

- Define a ***supersymmetric*** sigma model, Lagrangian and a ***supercharge*** for every symmetry.
- Extract a ***moduli space of instanto***n for the supersymmetric path integral associate.
- Clean this space (compactification. ...), and determine a ***local model*** by linearization. Deduce an ***expected dimension***.
- Compute ***invariants obtained as integrals*** over the moduli space.

Geometric engineering ?
One might think that for a long time, this machinery will be modeled as accurately as possible the world of elementary particles, and can adapt to new results that emerge from fundamental physics.

Références

[1] S.K.Donaldson, P.B. Kronheimer The geometry of Four -Manifolds, Clarendon Press - Oxford 1990.

[2] K.Hori, S. Katz, A. Klem, R. Pandharipande, R. Thomas, C. Vafa, R. Vakil, E. Zaslow,*Mirror symmetry*, AMS, 2003. Press.

[3] Edward Witten, Topological Quantum Field Theory, Commun. Math. Phys. 117, 353-386 (1988) K.Hori, S. Katz, A. Klem, R. Pandharipande, R. Thomas, C. Vafa, R. Vakil, E. Zaslow,*Mirror symmetry*, AMS, 2003. Press.