From QFT to 2-d supersymmetric TQFT: Autoduality, Mirror symmetry between maths and physics: Atiyah-Witten work: an "instantonic Geometry"

Ph Durand: *Conservatoire national des arts ét métiers Paris*

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Presentation

1. Introduction

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2. From Quantum field theory to topological fields theory.

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3. Lower dimensions, Four dimensional geometry.

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5. Rules for a "Geometric engineering" of the world ?

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I) Introduction

Atiyah Work

The introduction of supersymmetric field theories in two dimensions, has contributed to strengthen the links between geometry and physics.The beginning is ***Atiyah*** work, principally ***index theorem*** : a smooth form of ***Riemann-Roch*** and the applications on physics. With Segal he give axiom of ***topological quantum field theory***.

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BRST, cohomology

That helped extend the ***BRST formalism*** for supersymmetric topological field theories . Calculations of correlation functions in which the points are replaced by cohomology classes, lead to define spaces of instantons : ***moduli*** whose dimension calculated by the index theorem lead to enumerative geometry.

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Gromov and Witten

From 1985, Gromov have understood the importance of the concept of "curves" in symplectic geometry that products new instantons ***"holomorphics curves"*** point of departure of topological string theorie and mirror symmetry.

II) From QFT to TQFT

QFT

A quantum field theory can be seen in through a ***classical action*** was quantified from the ***path integral***. We can then define the ***correlation functions*** dependent on observable functions on selected points on the source space. These functions depend, for example, the metric that is chosen on the target space.

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TQFT

It appears natural to define a field theory "topological" free of sophisticated structures like differentiability, metric. In such a theory of correlation functions depend only on topological objects such as class (co)-homology

II) TQFT : From Atiyah-Segal program

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Cobordism

A TQFT of dimension $n+1$ associated to each n dimensional manifold X an vector space $V(X)$. ***A cobordism*** is data (M, X, Y), with $\partial M = (-X)\sqcup(Y)$ and $dimM = n+1$.

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Rules

TQFT must satisfy some axiom : Naturality, fonctoriality, normalization, multiplicativity, symmetry .

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$\langle \mathcal{O}_{\mu_1}, \mathcal{O}_{\mu_2}, ..., \mathcal{O}_{\mu_n} \rangle = \int [\mathcal{D}_{\phi} i] \mathcal{O}_{\mu_1}(\phi_i), \mathcal{O}_{\mu_2}(\phi_i), ..., \mathcal{O}_{\mu_n}(\phi_i) exp(-S(\phi))$
Topological means : $\frac{\delta}{\delta g_{\mu\nu}} \langle \mathcal{O}_{\mu_1}, \mathcal{O}_{\mu_2}, ..., \mathcal{O}_{\mu_n} \rangle = 0$

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Localization

The integral is ***localized*** on the space of instantons (zero-modes) and is zero for other configurations.

Localization in zero-dimensional supersymmetry

- A "Toy" model is given by taking map from space $\Sigma = \{P\}$ to target $M = \mathbb{R}$, the real line. In this context, a field is simply the variable x, the path integral is just $\mathcal{Z} = \int_M e^{-S(x)} dx$

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  $S(x, \psi_1, \psi_2) = \frac{h'(x)^2}{2} - h''(x)\psi_1\psi_2$.
  hence the partition function :
  $\mathcal{Z} = \int e^{\frac{-h'(x)^2}{2} + h''(x)\psi_1,\psi_2} dx d\psi_1 d\psi_2$
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Supersymmetric transformations

- For the example above, we can define supersymmetric transformations that respect this action.

  $\delta x = \epsilon_1 \psi_1 + \epsilon_2 \psi_2$

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Supersymmetric quantum mechanic

- Now we study a supersymmetric field theory in ***one dimension***. This is the model of ***supersymmetric quantum mechanics*** which has allowed Witten to give a new proof of the ***index theorem***. We considère the lagrangian :
  $L = \frac{\dot{x}^2}{2} - \frac{h'(x)^2}{2} + i(\bar{\psi}\dot{\psi} - \dot{\bar{\psi}}\psi) - h''(x)\bar{\psi}\psi$.
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- We can show :
  $\delta_\epsilon S = \int \delta L dt = \int \frac{d}{dt} L dt = 0$

Localization of supersymmetric quantum mechanic

- The two operators of supersymmetry, are associated ***supercharge*** Q, $\bar{Q}$ with $Q^2 = \bar{Q}^2 = 0$ and we deduce an elliptic complex :
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- $\frac{\partial}{\partial\beta} Tr(-1)^F e^{-\beta H} = -\int_{periodicBd} \mathcal{D}X\mathcal{D}\psi\mathcal{D}\overline{\psi} H e^{-S} = 0$

Limite : From 1-dim TQFT to 0-dim TQFT

The fundamental result is that ***only time-independent contribute*** : that reduce calculation to 0-dim TFT :
$\mathcal{Z} = Tr(-1)^F e^{-\beta H} = \sum_{h'(x_c)=0} \sqrt{\pi} \frac{h''(x_c)}{|h''(x_c)|}$

III) TQFT and Lower dimensional geometry

Lower dimensions

The geometry of one and two dimensions is well know. Henri Poincare invented algebraic topology, and the Betti numbers that generalizes the invariant Euler. It can be shown that in dimension smaller than three, the category of ***topological*** varieties is identical to that of ***differentiable*** manifolds. (1923 Kerékjarko for dimension 2, Moises, Bing (1950) for the dimension 3). However this result is false in dimension four (in 1956 Milnor shows that there are 27 structures on the differentiable sphere S_7.

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Two dimensions

In two dimensions, the classification of surfaces topological (so differentiable), is made by the ***Betti numbers***, and the ***orientability*** allows for example to distinguish between torus and Klein bottle. In particular, it can easily demonstrate the Poincare conjecture in dimension 2 : any simply connected compact surface ($\pi_1(S) = 0$) is homeomorphic to the sphere and as H_1 is abelianised of π_1, the first Betti number is zero and the surface does not contain holes.

III)Four dimensional geometry

Topological four manifolds

In four dimension, let X a compact a variety simply connected,we have $H_1(X) = 0$, so by Poincare duality, we also $H_3(X) = 0$, only $H_2(X)$ may be nonzero. One can intuit ***that two non-trivial cycles*** (areas), Σ_1, Σ_2 in general position in X ***contribute to the homology*** and thus provide topological information about it. is exactly what showed Freedman in 1982

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Intersection form

Four dimensionnal manifolds X, are classified by the ***intersection quadratic form*** $H_2(X, \mathbb{Z}) \times H_2(X, \mathbb{Z}) \to \mathbb{Z}$.
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Example

1. $I_{S^4} = 0$: there is not non trivial cycles in the four sphere $(H_2(S^4,\mathbb{Z}) = 0)$.
2. $I_{S^2\times S^2}, = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$: $H_2(S^2 \times S^2, \mathbb{Z}) = \mathbb{Z} \oplus \mathbb{Z}$: There is two cycles in general position $A = S^2 \times pt$, $B = pt \times S^2$ et $\langle A, B\rangle = \langle A, B\rangle = 1$, $\langle A, A\rangle = \langle B, B\rangle = 0$.
3. $I_{M\sharp N} = \begin{pmatrix} I_M & 0 \\ 0 & I_N \end{pmatrix}$:$H_2(M\sharp N,\mathbb{Z}) = H_2(M,\mathbb{Z}) \oplus H_2(M,\mathbb{Z})$, ($M\sharp N$ is connect sum of two four dimensional manifolds.

III) Four dimensional geometry

Theorem : Freedman (1982)

A simply connected four manifold X with even intersection form, q belongs to a ***unique homeomorphism class***, while if $q$is odd, there are just ***two non-homeomorphic*** X with q as their intersection form.

III) Four dimensional geometry

Theorem : Freedman (1982)

A simply connected four manifold X with even intersection form, q belongs to a ***unique homeomorphism class***, while if $q$is odd, there are just ***two non-homeomorphic*** X with q as their intersection form.

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Take $X = S^4$ and S^4_h an ***homotopic deformation*** of X, intersection form of X is zero so even. S^4_h has the same cohomology class than X by ***the previous theorem***, so the same intersection class that prove Poincaré theorem.

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Theorem : Donaldson (1983)

A simply connected smooth four manifold, with ***positive definite intersection form*** is always diagonalisable with the property : ***each eigenvalues equal to one***.

Yang-Mills : Donaldson theory

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- Donaldson define the ***moduli space*** of ASD connections as follows :
  $\mathcal{M}_{ASD} = \{[A] \in \mathcal{A}/\mathcal{G}/F_A^+ = 0\}$ usually in this theory $\mathcal{G} = SU(2)$

Local model for $\mathcal{M}_{ASD}$

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Donaldson invariants

- Let $\mathcal{A}^*$ irreductible connections, $\mathfrak{g}_E = adP$, $\mathcal{B}^* = \mathcal{A}^*/\mathcal{G}$
  We define a new space $\mathcal{B}^* \times X$, where we can compute invariants :

$$\begin{array}{ccc} \mathcal{A}^* \times_{\hat{\mathcal{G}}} \mathfrak{g}_E \\ \downarrow \pi \\ \mathcal{B}^* \times X \end{array} \qquad \begin{array}{ccc} \mathcal{A}^* \times \mathfrak{g}_E & \longrightarrow & \mathfrak{g}_E \\ \downarrow \pi & & \downarrow \pi \\ \mathcal{A}^* \times X & \longrightarrow & X \end{array}$$

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Transformation from Donaldson map

We have :

$$\begin{array}{lcl} x \in H_0(X) & \to & \mathcal{O}(x) \in H^4(\mathcal{M}_{ASD}) \\ \delta \in H_1(X) & \to & I_1(\delta) \in H^3(\mathcal{M}_{ASD}) \\ S \in H_2(X) & \to & I_2(S) \in H^2(\mathcal{M}_{ASD}) \end{array}$$

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From Witten to Donaldson

We have :
$I_1(\delta) = \int_{\delta \in H_1(X)} \mathcal{O}^{(1)}$
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IV) Ten dimension : String theory

TQFT

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Supersymmetric transformation preserving action

$$\delta x^I = \eta\chi^I \qquad \delta\chi^I = 0$$
$$\delta\psi_{\bar{z}}^i = \eta\partial_{\bar{z}}\phi_i \quad \delta\psi_z^{\bar{i}} = \eta\partial_z\overline{\phi_i}$$

If $\delta\psi_{\bar{z}}^i = \delta\psi_z^{\bar{i}} = 0$, we recognize the conditions of Cauchy-Riemann ! : The instantons of this model are curves "minimum energy" according to Gromov : holomorphic curves

(Virtual) dimension of space of holomorphic curves

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Dimension of $\mathcal{M}_g(X)$

By combining the two previous forms of the Riemann-Roch formula :
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- We could find directly this result in symplectic case (manifolds that are treated are Kähler, hence symplectic) relying on the index of a Fredholm operator for an elliptic complex adhoc.

Gromov-Witten invariant

- The notation [.] denote the fundamental class in $H_k(\overline{\mathcal{M}}_{g,n}(X,\beta),\mathbb{Q})$: We can now properly define the Gromov-Witten invariants. Indeed, if $[\omega_1],...,[\omega_n]$ are cohomology class in $H^*_{DR}(X)$ such that $\Sigma_{i=1}^n deg[\omega_i] = k$ "integration on moduli space " will then be a non-zero number, So we can expect count something.

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$$< [\omega_1],...[\omega_n] >_\beta = \int_{[\overline{\mathcal{M}}_{g,n}(X,\beta)]} ev^*([\omega_1]) \wedge ... \wedge ev^*([\omega_n])$$

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- This is completely analogous to how to obtain the invariants of Donaldson.

V) Conclusion

Set of rules for TQFT

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Geometric engineering ?

One might think that for a long time, this machinery will be modeled as accurately as possible the world of elementary particles, and can adapt to new results that emerge from fundamental physics.

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