Geometric, Algebraic and Topological Methods for Quantum Field Theory
Geometry of closed strings, A and B side of Witten
Part II: B side and mirror symmetry

Ph Durand: *Conservatoire national des arts et métiers Paris*

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Plan de l'exposé

1. Introduction

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5. Conclusion : Application to enumerative geometry.

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6. Références

I) Introduction

Mirror symmetry

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Mirror symmetry is a very accomplished form of ***T-duality*** appeared in bosonic theory which states that the partition function remains unchanged in the change $R \leftrightarrow \frac{1}{R}$, where R denotes the radius of compactification of extra dimensions.

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Variation of symplectic, complex structure

In a bosonic field theory, where the source space is the cylinder and target a torus, one can notice that the *T-duality* ***exchange*** symplectic structure deformation (area) and deformation of complex structure.

II) Landau Ginzburg model and deformation of complex manifolds

Correlation function on the B-side

At the A-side of the mirror , The correlation functions were calculated from holomorphic curves instantons , in this context, they will be calculated using the tools of complex geometry, the key point is the statistical physics and the Russian school of ***Arnold*** and his ***theory of singularity***.

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φ^4 theory and complex geometry

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Lagrangian of ferromagnetism

Starting from the Lagrangian in φ^4 :
$\mathcal{L}_{LG} = \partial_\mu\phi\partial_\mu\phi - V(T,\phi)$ where $V(T,\phi) = \frac{1}{4!}\lambda(T)\phi^4 + \frac{1}{2!}\mu^2(T)\phi^2$

Temperature, critical point

Critical temperature

At the critical temperature T_c "mass", $\mu^2(T_c) = 0$, so the correlation length (inverse mass) is infinite. At this temperature the field ϕ_0, solution of $\frac{\partial}{\partial\phi}V(T,\phi)$ is zero three times degenerated.

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Perturbation

A small perturbation $V(T_c, \phi) \to V(T_c, \phi) + \delta\mu^2(T)\phi^2$, ***solves*** the singularity and the is ***"symmetry breaking"***.

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Challenge

The challenge is to find a way of ***marginally perturb*** a potential theory in order to preserve the symmetry and defined by the fact a critical family of superpotentials.

Landau Ginzburg superpotential

Definition

The superpotential is an holomorphic function $\mathcal{W} : \mathbb{C}^M \to \mathbb{C}$ is chosen as a potential $V(x) = \sum_1^N |\partial_i \mathcal{W}(X)|^2 = \sum_1^N \partial_i \mathcal{W}(X) \partial_i \mathcal{W}(X)^*$ if we consider only one field, one can consider the function $\mathcal{W}(X) = \frac{1}{(n+1)!} X^{n+1}$

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Lagrangian LG-supersymmetric

The bosonic part of the supersymmetric Lagrangian is written then :
$\mathcal{L}_{N=2}^{LG} = -\partial_+ X^* \partial_- X - \partial_- X^* \partial_+ X + V(X)$
where, $\partial_\pm$ denote the light-cone coordinates.

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Extrema of the potential

There is : $V(X) = 0 \Leftrightarrow \partial_i \mathcal{W}(X_0) = 0$, so it is relevant to define the ***Chiral ring*** $\mathcal{R}_{\mathcal{W}} = \dfrac{\mathbb{C}[X]}{\partial \mathcal{W}(X)}$ where the ratio is proportional to the polynomials of $\partial_i \mathcal{W}(X) : P(X) = P^i(X) \partial_i \mathcal{W}(X)$

Singularity theory and marginal deformations

Deformed superpotential

Deformations respecting the Chiral ring are given by : $\mathcal{W}_{def}(X) = \mathcal{W}(X) + \sum_{P \in \mathcal{R}_{\mathcal{W}}} t_P P(X)$

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Example

If we choose $\mathcal{W}(X,Y,Z) = \frac{1}{3}(X^3 + Y^3 + Z^3)$, the deformed potential is given by $\mathcal{W}_{def}(X,Y,Z) = \mathcal{W}(X,Y,Z) + t_0 + t_1 X + t_2 Y + t_3 Z + t_4 XY + t_5 YZ + t_6 ZX + t_7 XYZ$

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Marginal deformations

Only non vanishing term $\mu = t_7$ preserves the critical situation, it does not break the $\mathbb{Z}_3$ symmetry $(X, Y, Z) \to (exp(\frac{2ki\pi}{3})X, exp(\frac{2ki\pi}{3})Y, exp(\frac{2ki\pi}{3})Z)$ we just define a ***continuous*** family of ***allowed*** perturbations :

$$\mathcal{W}_{def}(X, Y, Z, \mu) = \frac{1}{3}(X^3 + Y^3 + Z^3) + \mu XYZ$$

III) Calabi-Yau deformation theory

Superpotential and Calabi-Yau

The hypersurface of a complex projective space obtained by canceling $\mathcal{W}_{def}(X, Y, Z, \mu) = \frac{1}{3}(X^3 + Y^3 + Z^3) + \mu XYZ$ is the the simplest example of Calabi-Yau. Is an elliptic curve or complexe torus .

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Hypersurfaces

There is a strong constraint between the degree of a hypersurface and the dimension of the ambient space.

Example of Calabi-Yau

Exact sequence of hypersurface

Let us write the exact sequence associated to a hypersurface of degree d :
$0 \to \mathcal{O}_{\mathbb{P}^{n+1}}(-d) \to \mathcal{O}_{\mathbb{P}^{n+1}} \to \mathcal{O}_X \to 0$
With exact long sequence in cohomology, one can calculate the cohomology groups associated.

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Result

$H^n(X, \mathcal{O}_X) = \mathbb{C}^{C_{n-1}^{d-1}} = \mathbb{C}$ (***Calabi-Yau conditions***) Necessary $d = n + 2$

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Examples

- $n = 1$ (d = 3) : ***Elliptic Curves***
- $n = 2$ (d = 4) : ***K3 Surfaces***
- $n = 3$ (d = 5) : ***Quintic threefold***

Calabi Yau manifold

Deformations

Two types

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Kähler Déformation

It can also vary the Kähler structure, referenced by $H^{1,1}(X) = H^1(X, \Omega^1_X)$

IV) Mirror symmetry of closed strings

Mirror symmetry

There are two supersymmetric fields theories (**CFT**) in duality witch satisfies : $h^{2,1}(X) = h^{1,1}(MX)$ and $h^{1,1}(X) = h^{2,1}(MX)$: deform the complex structure of M amounts to deform the volume of his mirror.

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Hodge diamond

Hodge diamond plots the Hodge numbers of a complex manifold
Hodge diamond of Calabi-Yau (Quintic Threefold)

$$\begin{array}{ccccccc}
 & & & \mathrm{h}^{0,0}=1 & & & \\
 & & \mathrm{h}^{1,0}=0 & & \mathrm{h}^{0,1}=0 & & \\
 & \mathrm{h}^{2,0}=0 & & \underline{\mathrm{h}^{1,1}=1} & & \mathrm{h}^{0,2}=0 & \\
\mathrm{h}^{3,0}=1 & & \underline{\mathrm{h}^{2,1}=101} & & \mathrm{h}^{1,2}=101 & & \mathrm{h}^{0,3}=1 \\
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"B side" physical origin

A side

On the A Side, supersymmetric constraints lead to what the action does ***depends only on the the Kähler form*** ; instantons are holomorphic curves. The calculation of correlation functions is difficult because it takes into account correction on the degree curves (***invariants of Gromov-Witten***)

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B side

On the side "B", $BRST$ formalism explained in the other side of the mirror applies here : instantons are ***constants maps*** from the world-sheet Σ on a point of target space X. The correlation functions are simpler to calculate : they require no ***instanton correction***. If X is a Calabi-Yau 3, the 3 points correlation function is :

$\langle W_A W_B W_C \rangle = \int_X \Omega^{jkl} A_j \wedge B_k \wedge C_l \wedge \Omega$

A, B, C belong to $H^1(X, TX)$ and ***depend on the complex structure***, Ω is (3.0) top-form ***holomorphic***.

Calculus of correlation functions

Mirror symmetry Principe

The two numbers $\underline{h^{1,1}(X){=}1}$ and $\underline{h^{2,1}(X) = 101}$, count the number of deformation structures respectively Kähler and complex. The principle of mirror symmetry, gives $\underline{h^{1,1}(MX){=}101}$ and $\underline{h^{2,1}(MX) = 1}$. He said in addition that ***correlation functions*** calculated from both sides of the mirror are ***identical***.

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Application

In physic If a problem is difficult at the A side, we can try to solve it at the B side. ***In mathematics*** passing through the mirror application can solves ***so important*** old problem of ***Enumerative Geometry***

The Quintic and its mirror

Quintic threefold

Recall that the ***homogeneous quintic*** in $\mathbb{P}^4$, is obtained by canceling the superpotential : $\mathcal{W} = \frac{1}{5}(X_0^5 + ... + X_4^5)$

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Miroir du quintique

The ***mirror quintic variety*** is associated with a ***crepant*** resolution of :

$$\{(X_0, ..., X_4) \in \mathbb{P}^4 / \frac{1}{5}(X_0^5 + ... + X_4^5) - \mu X_0 ... X_4 = 0\}_{/G}$$

with $G = \{(a_0, ..., a_4) \in \mathbb{Z}_{/5} / \sum a_i = 0\}_{/\mathbb{Z}_{/5} = \{(a,a,a,a,a)\}} \simeq (\mathbb{Z}_{/5})^3$
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localization

The overall construction of the ***mirror map*** is difficult. Should be localize and built the map at the ***neighborhood of a point***.

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Mathematical Tools

We will briefly describe in the following ***useful mathematics*** for the B-side of the quintic ;
We can deduce predictions on the number of rational curves in the quintic treefold.

Mirror-map

One parameter family

The principle of mirror symmetry says that $< H, H, H >=< \theta, \theta, \theta >$, if tH denotes an ***curve parameter*** in the module of Kähler X we set $H = \frac{d}{dt} = 2\pi iq\frac{d}{dq}$ its tangent vector ($q = exp(2\pi it)$ Local coordinates for this module.

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Correspondance

Then we can write $H = 2\pi iq\frac{d}{dq}) \leftrightarrow \theta = 2\pi iq\frac{dx}{dq}\frac{d}{dx}$

$< H, H, H >= (2\pi iq\frac{d}{dq})^3 < \frac{d}{dx}, \frac{d}{dx}, \frac{d}{dx} >= (2\pi i\frac{q}{x}\frac{dx}{dq})^3 < x\frac{d}{dx}, x\frac{d}{dx}, x\frac{d}{dx} >$

the right part of the last term is : $< \theta, \theta, \theta >$

Mathematical Tools

Elliptic curves

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Monodromy

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Residue map

Residue map : We can generalize the formula for residues of a complex variables function around $z = 0$ by replacing function ***differential forms*** and point by ***hypersurface***. This will be very useful for calculating periods.

Elliptic curves

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- The one parameter family of deformations of an elliptic curve is : $X^3 + Y^3 + Z^3 - 3\psi XYZ = 0$
- If α and β are ***homology cycles*** , they depend then ψ, we can find τ to from ***ratio periods*** : $\int_\alpha \Omega$, $\int_\beta \Omega$.
  Solving a ***differential equation*** called ***Picard-Fuchs***

Family of elliptic curves, monodromy

Degeneration of a family of curves

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Monodromy theorem

Either X_t where t varies along a loop in $\pi_1(D^2 - \{0\}, t_0)$ around 0, (***Wilson loop***), all elements of this family are ***diffeomorphic*** .This induces an automorphism on homology : $\varphi_\star \in \mathcal{A}ut(H_n(X_{t_0}), \mathbb{Z}))$

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Pratical example

Let's illustrate by taking a one parameter family of ***elliptic curve*** $C_\lambda = \{(Y^2Z = X^3 + X^2Z - \lambda Z^3\} \subset \mathbb{CP}^2$ which is expressed in affine coordinates : $C_\lambda : y^2 = x^3 + x^2 - \lambda$: elliptic curve defined by an algebraic equation.

Family of elliptic curves, monodromy

- The parameter λ of the ***elliptic curve*** is the signature of the variation of complex structure, the geometric expression is $E_\tau = \mathbb{C}/(1, \tau(\lambda))$ when t revolves around the origin $\tau(\lambda) \to \tau(\lambda) + 1$ with τ function of : λ : $\tau(\lambda) = \frac{ln\lambda}{2\pi i}$

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- We find that the complex structure varies but the symplectic structure is unchanged.
- We justify the choice of new coordinate $q(\lambda) = e^{2\pi i \tau(\lambda)}$, because a passage to the limit gives : when $\lambda \to 0$, $Im\tau(\lambda) \to +\infty$, so $q(\lambda)$, where $q(\lambda)$ is an holomorphic function of λ which also tends to 0 ; so it is a local coordinate for this family of elliptic curves. We spoke in physics : ***large complex structure limit(LCSL)*** in "bijection" with ***large volume limit (LVL)*** on the A side

Application to the quintic

- Is the ***quintic*** $\sum_{i=0}^{4} x_i^5 - \psi x_0 x_1 x_2 x_3 x_4 = 0$. As with elliptic curves, Morrison showed by standing near $x = (\frac{1}{\psi})^5 = 0$ we could find t : (deformation side Kähler) from ψ (or x).

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- As with elliptic curves, $t = \frac{\int_{\gamma_1} \Omega}{\int_{\gamma_0} \Omega}$, we must calculate $\phi_i(x) = \int_{\gamma_i} \Omega$, $i = 0, 1$. γ_0 is invariant under monodromie around $x = 0$,(***large complex structure limit (LCSL)***), $\psi \to \infty$ $\gamma_1 \to \gamma_0 + \gamma_1$. These two quantities depend on ψ (or x)

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- Finally, we find :
  $\phi_0(x) = \sum_{n=0}^{\infty} \frac{5n!}{(n!)^5} x^n$,
  $\phi_1(x)$=$\phi_0(x) log(x) + f(x)$, with $f(x) = 5 \sum_{n=0}^{\infty} \frac{5n!}{(n!)^5} (\sum_{j=n+1}^{5n} \frac{1}{j}) x^n$

Calculation of Yukawa couplings on B side

- Let $\Theta^{(i)} = (x\frac{d}{dx})^{(i)}$, the equation of Picard-Fuch written :
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- By identification,
  $< H, H, H > = (2\pi i\frac{q}{x}\frac{dx}{dq})^3 < x\frac{d}{dx}, x\frac{d}{dx}, x\frac{d}{dx} > = \frac{c_2(2\pi i\frac{q}{x}\frac{dx}{dq})^3}{(1+5^5x)\phi_0(x)^2}$

V Conclusions : Application to enumerative geometry

- The parameter t Kähler deformation, expressed as a function of the ratio of the first two periods, we get : $q = e^{2i\pi \frac{\phi_1(x)}{\phi_0(x)}}$ Where : $q = c_1(x - 770x^2 + ...)$ and conversely $x = \frac{q}{c_1} + 770(\frac{q}{c_1})^2 + ...$

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- We can now calculate everything according to the variable q : $< H, H, H >= (2\pi i)^3(-c_2 - 575(\frac{c_2}{c_1})q - 19575(\frac{c_2}{c_1^2})q^2 + \ldots)$ It remains to calculate the constants c_1 and c_2.

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- Finally, we can enumerate the ***number curve of a rational quintic*** of $\mathbb{P}^4$ all degrees
  $< H, H, H >= 5 + \sum_{d=1}^{\infty} n_d \frac{d^3 q^d}{1-q^d} = \mathbf{5} + \mathbf{2875}\frac{q}{1-q} + \mathbf{609250}.2^3\frac{q^2}{1-q^2} + ...$

VII) Références

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